QUANTUM HOMOGENEOUS SPACES WITH FAITHFULLY FLAT MODULE STRUCTURES

BY

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ABSTRACT

Let A be a Hopf algebra with bijective antipode and $B \subset A$ a right coideal subalgebra of A. Formally, the inclusion $B \subset A$ defines a quotient map $G \to X$ where G is a quantum group and X a right homogeneous G-space. From an algebraic point of view the G-space X only has good properties if A is left (or right) faithfully flat as a module over B.

In the last few years many interesting examples of quantum G-spaces for concrete quantum groups G have been constructed by Podleś, Noumi, Dijkhuizen and others (as analogs of classical compact symmetric spaces). In these examples B consists of infinitesimal invariants of the function algebra A of the quantum group. As a consequence of a general theorem we show that in all these cases A as a left or right B-module is faithfully flat. Moreover, the coalgebra A/AB^+ is cosemisimple.

0. Introduction

Let A be a Hopf algebra with bijective antipode over the ground field k, and $B \subset A$ a right coideal subalgebra, that is a subalgebra with $\Delta(B) \subset B \otimes A$. We can think of the inclusion $B \subset A$ as defining a quotient map $G \to X$ where G is a quantum group and X is a quantum space with right G-action or a right G-space.

Since A is not commutative in general, A/AB^+ (recall that $B^+ = \mathrm{Ker}(\varepsilon|_B)$ is the augmentation ideal of B) is just a coalgebra and a left A-module but not

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a Hopf algebra. Thus X usually is not the quotient of G by some quantum subgroup. But if A is B-faithfully flat one still gets B back from the quotient map $A \to A/AB^+$ as the A/AB^+-coinvariant elements of A.

From an algebraic point of view [T1, Sch, M1] the inclusion $B \subset A$ only has good properties if A is faithfully flat as a left (or right) module over B.

In the last few years many interesting examples of quantum G-spaces for concrete quantum groups G have been constructed. Podleś [P] found a continuously parametrized family of $\mathrm{SU}_q(2)$-spaces which are analogs of the classical 2-sphere $\mathrm{SU}(2)/\mathrm{SO}(2)$. Dijkhuizen and Noumi [DN] defined more generally a family of $U_q(n)$-spaces called quantum projective spaces. [D] gives a survey on other quantum G-spaces which are analogs of classical compact symmetric spaces such as $\mathrm{SU}(n)/\mathrm{SO}(n)$ or $\mathrm{SU}(2n)/\mathrm{Sp}(n)$. In all these cases, the right coideal subalgebra B is defined by infinitesimal invariants with respect to a coideal I in U, the U-part of the quantum group G (the classical counterpart of U is the universal enveloping algebra of the Lie algebra of the algebraic group). Moreover, B is a $*$-subalgebra of the Hopf $*$-algebra A which is guaranteed by the natural condition $S(I)^* \subset I$.

In this paper we show as a consequence of the general theorem 2.2 that in the above examples, the extension $B \subset A$ has the crucial property of faithful flatness.

Note that the module structure of the quantized universal enveloping algebra U over its right coideal subalgebras is much easier to investigate, because U is pointed (that is all its simple subcoalgebras are one-dimensional) whence the module structure is faithfully flat whenever the set of group-likes in the right coideal subalgebra is a group [M2]. In particular this condition holds for the new examples in [L].

Let U be a Hopf algebra with bijective antipode, $K \subset U$ a left coideal subalgebra and $\mathcal{C}$ a tensor category of finite dimensional left U-modules (for example modules of type 1 for $U_q(\mathfrak{g})$, where $\mathfrak{g}$ is a semisimple complex Lie algebra and $q \in k$ not a root of 1). Let $A = U^0_{\mathcal{C}}$ be the Hopf dual of U with respect to $\mathcal{C}$ and

$$B := \{a \in A \mid a \cdot K^+ = 0\},$$

the algebra of infinitesimal invariants (here "$\cdot$" denotes the natural (U,U)-bimodule structure on $A \subset U^*$, as recalled in the beginning of section 2). Note that B is a subalgebra since K^+ is a coideal of U and A is a right (and left) K-module algebra via "$\cdot$". It is easy to see that $B \subset A$ is a right coideal subalgebra. But in general B is not a Hopf subalgebra. In this set-up we now assume that all modules in $\mathcal{C}$ are semisimple over K. Then we call K $\mathcal{C}$-semisimple. We show in Theorem 2.2 that

- A is faithfully flat as a left and right B-module, more precisely A as a left and right B-module is a direct sum of finitely generated and projective B-modules with one direct summand being B.
- The quotient coalgebra A/AB^+ is cosemisimple, and if K is commutative and k is algebraically closed, then A/AB^+ is spanned by group-like elements.

In particular, in the case of Podleś' quantum spheres, A/AB^+ is spanned by group-like elements. This answers a question of Brzeziński [B]. Recall that a coalgebra C is **cosemisimple** [Sw] if C is the direct sum of its simple subcoalgebras. C is spanned by group-like elements if and only if C is cosemisimple and pointed.

In section 3 we show that K in fact is $\mathcal{C}$-semisimple in many important cases. Let U be a Hopf $*$-algebra, and assume that U is pointed (this holds for all the quantized universal enveloping algebras $U_q(\mathfrak{g})$). If $I \subset U$ is a coideal with $S(I)^* \subset I$ and $B := \{a \in A |\ a \cdot I = 0\}$ as in the examples described above, then $IU = K^+U$ where K is the left coideal subalgebra of the right U/IU-coinvariant elements in U. Hence we may also write $B = \{a \in A|\ a \cdot K^+ = 0\}$ as in Theorem 2.2. If we assume that all modules $V \in \mathcal{C}$ have a hermitian inner product $\langle , \rangle$ such that $\langle xv, w\rangle = \langle v, x^*w\rangle$ for all $v, w \in V$ and $x \in U$, then by Corollary 3.3, $K = K^*$ is $\mathcal{C}$-semisimple, and our abstract Theorem 2.2 applies.

In sections 4 and 5 we consider in $U := U_q(\mathrm{sl}(2))$ and $U_q(\mathfrak{g})$, for $\mathfrak{g}$ a semisimple complex Lie algebra and $q \in k$ not a root of unity, an arbitrary skew-primitive element x with $\Delta(x) = g \otimes x + x \otimes 1$, where g is a group-like element in U. Then the subalgebra $k[x]$ generated by x is a left coideal subalgebra in U. We take for $\mathcal{C}$ the class of finite dimensional representations of type 1 and define

$$B := \{a \in A|\ a \cdot x = 0\}$$

in $A := U^0_{\mathcal{C}}$, the q-deformed function algebra of the connected, simply connected algebraic group with Lie algebra $\mathfrak{g}$. Somewhat surprisingly we show in Theorem 5.2 that for all x (up to one case) the following conditions are equivalent:

(a) $k[x]$ is $\mathcal{C}$-semisimple, that is x acts on all finite dimensional U-modules as a diagonalizable operator.

(b) A is left or right faithfully flat over B.

(c) A/AB^+ is spanned by group-like elements.

Moreover, condition (a) is equivalent to an explicit numerical condition on the coefficients of x.

Thus we see that condition (a) which was studied by Noumi and Mimachi

[NM] in connection with Podleś' quantum spheres is equivalent to the abstract algebraic condition of faithful flatness in (b).

Most calculations in section 4 and special versions of some arguments in the proof of 2.4(1) are already contained in the first author's diploma thesis [Mü].

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1. Preliminaries and some general results

In the following we fix a field k which is the ground field for all algebras and vector spaces. For definitions and basic results on Hopf algebras see [Sw, M]. We use the simplified notation $\Delta(x) = \sum x_1 \otimes x_2$ for the coproduct in a coalgebra. Let V be a vector space. In the following (except for section 3), we write V^* for the dual space $\mathrm{Hom}(V, k)$ of V.

To study sub- and quotient objects of Hopf algebras it is crucial to consider faithfully flat modules and faithfully coflat comodules. Recall that a right module M over an algebra B is called **flat** respectively **faithfully flat** if and only if the functor $M \otimes_B -\colon {}_B\mathcal{M} \to {}_k\mathcal{M}$ from the category ${}_B\mathcal{M}$ of left B-modules to the category ${}_k\mathcal{M}$ of k-vector spaces preserves respectively preserves and reflects exact sequences. Dually, a right comodule V over a coalgebra C is called **coflat** respectively **faithfully coflat** if and only if the cotensor product $V\square_C -: {}^C\mathcal{M} \to {}_k\mathcal{M}$ preserves respectively preserves and reflects exact sequences. Here, ${}^C\mathcal{M}$ is the category of left C-comoduless, and if W is a left C-comodule, then the **cotensor product** is defined as the kernel of

$$\Delta_V \otimes \mathrm{id} - \mathrm{id} \otimes \Delta_W\colon V \otimes W \to V \otimes C \otimes W,$$

where Δ_V and Δ_W are the comodule structure maps of V and W. Of course, these notions are defined in the same way for modules or comodules on the other side.

Let A be a Hopf algebra. If $I \subset A$ is a vector subspace with quotient map $\pi\colon A \to A/I$ (usually I will be a coideal and a left or right ideal), we let

$${}^{\mathrm{co}A/I}A := \{a \in A \mid \sum \pi(a_1) \otimes a_2 = \pi(1) \otimes a\}$$

be the set of left A/I**-coinvariant elements**. If $I \subset A$ is a coideal, then the quotient map $\pi\colon A \to A/I$ is a coalgebra map, and A is a right (respectively left) A/I-comodule via π with comodule structure $(\pi \otimes \mathrm{id})\Delta$ (respectively

$(\mathrm{id}\otimes\pi)\Delta)$. Dually for a subalgebra B of A we consider A as a left and a right B-module by restricting the multiplication in A. A **right** (respectively **left**) **coideal subalgebra** B of A is a subalgebra $B\subset A$ with $\Delta(B)\subset B\otimes A$ (respectively $A\otimes B$).

THEOREM 1.1 ([T1, Theorem 2]): *Let A be a Hopf algebra and $I\subset A$ a coideal and a left ideal with quotient map $\pi\colon A\to A/I$. Define $B:={}^{\mathrm{co}A/I}A$. Assume A is faithfully coflat as a right A/I-comodule via π. Then $B\subset A$ is a right coideal subalgebra, $I=AB^+$, where $B^+=\mathrm{Ker}(\varepsilon|_B)$ is the augmentation ideal of B, and A is faithfully flat as a right B-module.*

Moreover, it is shown in [T1] that $M\mapsto A\otimes_B M$ is an equivalence between ${}_B\mathcal{M}$ and the category of left $(A/I,A)$-Hopf modules.

Later on we will need the categories $\mathcal{M}_B^A$ and ${}_B\mathcal{M}^A$ of (A,B)-**Hopf modules** for a right coideal subalgebra B of a Hopf algebra A. Objects in $\mathcal{M}_B^A$ (respectively ${}_B\mathcal{M}^A$) are right (respectively left) B-modules V which are right A-comodules such that the comodule structure map $\Delta_V\colon V\to V\otimes A$ is right (respectively left) B-linear, where $V\otimes A$ is a right (respectively left) B-module via $(v\otimes a)b:=\sum vb_1\otimes ab_2$ (respectively $b(v\otimes a)=\sum b_1v\otimes b_2a$) for all $v\in V$, $a\in A$, and $b\in B$. Morphisms are right A-colinear and right (respectively left) B-linear maps. Note that $B\in\mathcal{M}_B^A$ and $B\in{}_B\mathcal{M}^A$ where the restriction of $\Delta\colon A\to A\otimes A$ is the comodule structure.

THEOREM 1.2 [MW, 2.1]: *Let A be a Hopf algebra with bijective antipode and $B\subset A$ a right coideal subalgebra. Let $\bar{A}:=A/AB^+$ with quotient map $\pi\colon A\to\bar{A}$. Then $\bar{A}$ is a quotient coalgebra and a quotient left A-module of A, and the following are equivalent:*

(1) *A is faithfully coflat as a left $\bar{A}$-comodule via π, and $B={}^{\mathrm{co}\bar{A}}A$.*
(2) *A is faithfully flat as a B-module.*
(3) *A is projective as a left B-module, and B is a left B-direct summand in A.*
(4) *A is flat as a left B-module, and B is a simple object in $\mathcal{M}_B^A$.*
(5) *The functor $\mathcal{M}_B^A\to\mathcal{M}^{\bar{A}}$, $M\mapsto M/MB^+$ is an equivalence.*

Proof: This is [MW, 2.1] when (3) is replaced by

(3′) A is a projective generator as a left B-module.

We have to show the equivalence of (3) and (3′). Assume (3′). Then A is projective and faithfully flat as a left B-module by [MW]. Hence B is a left B-direct summand in A by [R, 2.11.29]. ∎

The simplicity condition in (4) means that any non-zero right B-submodule and right A-subcomodule of B is equal to B.

The categorical characterization (5) shows the significance of the equivalent conditions in 1.2. Moreover, by [M1, 1.11] the mapping

$$\left\{ B \,\middle|\, \begin{array}{l} B \subset A \text{ is a right coideal} \\ \text{subalgebra, } A \text{ is left faith-} \\ \text{fully flat over } B \end{array} \right\} \to \left\{ I \,\middle|\, \begin{array}{l} I \subset A \text{ is a coideal and left} \\ \text{ideal and } A \text{ is left faithfully} \\ \text{coflat over } A/I \end{array} \right\}$$

is a bijection between sub- and quotient objects of A satisfying the corresponding conditions in 1.2.

Remark 1.3:

(1) If we apply 1.2 to the dual algebras $B^{\mathrm{op}} \subset A^{\mathrm{op}}$ (A^{op} is a Hopf algebra since the antipode of A is bijective), we get the dual theorem where $\bar{A}$ is now A/B^+A, $\mathcal{M}_B^A$ is replaced by ${}_B\mathcal{M}^A$, and A is considered as a right B-module.

(2) Let A be a Hopf algebra and $B \subset A$ a right coideal subalgebra. We note the following simplicity criterion: If B is a left B-direct summand in A then B is simple in $\mathcal{M}_B^A$.

Proof: Let $f\colon A \to B$ be a left B-linear map such that $f|_B = \mathrm{id}$. Let $X \subset B$ be a non-zero subobject in $\mathcal{M}_B^A$. Then XA is a non-zero Hopf module in $\mathcal{M}_A^A$. Since $\mathcal{M}_A^A \cong {}_k\mathcal{M}$ by the fundamental theorem of Hopf modules [Sw, 4.1.1], A is simple in $\mathcal{M}_A^A$ and $XA = A$. Hence there exist finitely many elements $x_i \in X$, $a_i \in A$ such that $\sum_i x_i a_i = 1$. Then $1 = f(1) = \sum_i x_i f(a_i) \in X$, and $X = B$. ∎

For completeness we give the short proof of the following important observation of Koppinen.

LEMMA 1.4: [Ko] *Let A be a Hopf algebra with bijective antipode S and $B \subset A$ a right (resp. left) coideal with $1 \in B$. Then $S(AB^+) = B^+A$ (resp. $S(B^+A) = AB^+$).*

Proof: We assume that B is a right coideal (and then apply the result to the dual coalgebra of A to get the lemma for left coideals). Let $\{x_j \mid j \in J\}$ be a basis of B^+. Then for all $x \in B$, $\Delta(x) = 1 \otimes x + \sum_j x_j \otimes y_j$ for some $y_j \in A$. Now assume $x \in B^+$ and apply $\mu \circ (S \otimes \mathrm{id})$ and $\mu \circ (\mathrm{id} \otimes S)$, where μ denotes multiplication in A:

$$0 = 1\varepsilon(x) = x + \sum_j S(x_j)y_j, \quad \text{and} \quad 0 = S(x) + \sum_j x_j S(y_j).$$

Therefore, $B^+ \subset S(B^+)A$ and $S(B^+) \subset B^+A$. Hence $B^+A = S(B^+)A$ and $S(AB^+) = S(B^+)A = B^+A$. ∎

COROLLARY 1.5: *Let A be a Hopf algebra with bijective antipode and $I \subset A$ a coideal and left ideal with quotient map $\pi\colon A \to A/I$. Assume A/I is cosemisimple, and let*

$$A/I = \bigoplus_{j\in J} C_j$$

be the direct sum of the simple subcoalgebras C_j for $j \in J$, of A/I. Then:

(1) *$I = AB^+$, and A is left and right faithfully flat over B and left and right faithfully coflat over A/I via π.*

(2) *For all $j \in J$, let*

$$_jA := \{a \in A \mid \sum \pi(a_1) \otimes a_2 \in C_j \otimes A\},$$
$$A_j := \{a \in A \mid \sum a_1 \otimes \pi(a_2) \in A \otimes C_j\}.$$

Then $A = \bigoplus_{j\in J} {}_jA = \bigoplus_{j\in J} S(A_j)$, and for all j, ${}_jA$ respectively $S(A_j)$ is a right coideal in A and a finitely generated and projective right respectively left module over B. If $1 \in J$ is the distinguished index with $C_1 = k\pi(1)$, then $B = {}_1A = S(A_1)$.

Proof: (1) By [Sch, 1.3], A is right faithfully coflat over A/I if and only if

(a) A is right coflat over A/I and

(b) π splits as a map of right A/I-comodules.

Since A/I is cosemisimple, any exact sequence of right A/I-comodules splits. In particular, (a) and (b) hold. Thus we see that A is right and by the same argument left faithfully coflat over A/I. Since A is right faithfully coflat over A/I, we get from 1.1 that $AB^+ = I$ and A is right faithfully flat over B. Then $B \subset A$ is a right coideal subalgebra, $B = {}^{\mathrm{co}A/AB^+}A$ and A is left faithfully coflat over A/AB^+. Hence the equivalent conditions in 1.2 hold, and A is left faithfully flat over B, too.

(2) By (1), B is a right coideal subalgebra of A, A is left and right faithfully flat over B and $\bar{A} = A/AB^+$ is cosemisimple. Hence by 1.2,

$$\mathcal{M}_B^A \to \mathcal{M}^{\bar{A}},\ M \mapsto M/MB^+, \quad \text{and} \quad \mathcal{M}^{\bar{A}} \to \mathcal{M}_B^A,\ V \mapsto V\square_{\bar{A}}A,$$

are quasi-inverse category equivalences. For any right $\bar{A}$-comodule V, the Hopf module structure on $V\square_{\bar{A}}A$ is given by multiplication and comultiplication on A. Since $\bar{A} = \bigoplus_j C_j$ is a decomposition of right $\bar{A}$-comodules, $\bar{A}\square_{\bar{A}}A \cong \bigoplus_j C_j\square_{\bar{A}}A$

as right A-comodules and right B-modules. Moreover, for all j the natural isomorphism

$$A \xrightarrow{\cong} \bar{A}\square_{\bar{A}} A, \quad A \mapsto \sum \pi(a_1) \otimes a_2,$$

maps ${}_jA$ onto $C_j \square_A A$. Hence $A = \bigoplus_j {}_jA$ is a decomposition in $\mathcal{M}_B^A$. By construction, ${}_1A = {}^{\mathrm{co}\bar{A}}A = B$. All the ${}_jA$ are projective right B-modules since A is projective as a right B-module by 1.3(1). They are finitely generated over B since ${}_jA \cong C_j \square_{\bar{A}} A$ and C_j is finite dimensional. More generally, let V be any finite dimensional right $\bar{A}$-comodule. Then $M = V \square_{\bar{A}} A$ is finitely generated as a right B-module. To see this write M as the ascending union of all Hopf submodules XB where X is a finite dimensional right A-subcomodule of M. Let $F\colon \mathcal{M}_D^A \xrightarrow{\cong} \mathcal{M}^{\bar{A}}$ be the category equivalence of 1.2. Then $F(M) \cong V$ is the ascending union of all $F(XB)$. Since V is finite dimensional, $F(M) = F(XB)$ for some X, hence $M = XB$ is B-finitely generated.

To get the decomposition of left B-modules we apply the previous result to A^{op}. Then $B^{\mathrm{op}} \subset A^{\mathrm{op}}$ is a right coideal subalgebra and A^{op} is left and right faithfully flat over B^{op}. By Koppinen's lemma 1.4, $S(AB^+) = B^+A$ and

$$\sigma\colon A/AB^+ \xrightarrow{\cong} A/B^+A, \ \bar{a} \mapsto \overline{S(a)},$$

is a coalgebra antiisomorphism. Therefore $A/B^+A = \bigoplus_j \sigma(C_j)$ is a direct sum of simple subcoalgebras and $A^{\mathrm{op}}/A^{\mathrm{op}}(B^{\mathrm{op}})^+ = A/B^+A$ is cosemisimple. Thus we know from the previous proof that $A = \bigoplus_j {}_j\tilde{A}$ where for all j,

$${}_j\tilde{A} := \{a \in A \mid \sum \tilde{\pi}(a_1) \otimes a_2 \in \sigma(C_j) \otimes A\}$$

is a finitely generated projective left B-module and a right coideal, and ${}_1\tilde{A} = B$. Here $\tilde{\pi}\colon A \to A/B^+A$ is the canonical map. Finally, for all j, ${}_j\tilde{A} = S(A_j)$ since for all $a \in A$,

$$\begin{aligned} a \in {}_j\tilde{A} &\iff \sum \tilde{\pi}(a_1) \otimes a_2 \in \sigma(C_j) \otimes A \\ &\iff \sum \pi(S^{-1}(a_1)) \otimes S^{-1}(a_2) \in C_j \otimes A \\ &\iff S^{-1}(a) \in A_j. \quad \blacksquare \end{aligned}$$

2. A class of homogeneous spaces defined by infinitesimal invariants

We first collect some well-known results and notations on duality (cf. [M, Chapter 9], [J, I.1.4], [T2, section 1]). Let U be an algebra. The **dual coalgebra** $U^0 \subset U^*$ is spanned by the matrix coefficients of all finite dimensional left U-modules V.

If ρ: $U \to \mathrm{End}(V)$ is the representation of U, C^V denotes the image of the dual coalgebra $(\mathrm{End}(V))^*$ under ρ^*. Thus C^V is the k-linear span of all **matrix coefficients** $c_{f,v} \in U^*$, $f \in V^*$, $v \in V$, where $c_{f,v}(u) := f(uv)$ for all $u \in U$. If (v_i), (f_i) are dual bases of V, V^*, the coalgebra structure of C^V is explicitly given by $\Delta(c_{f,v}) = \sum_i c_{f,v_i} \otimes c_{f_i,v}$ for all $f \in V^*$, $v \in V$. Then U^0 is the sum of all the subcoalgebras C^V, and $C^{V_1 \oplus V_2} = C^{V_1} + C^{V_2}$ for finite dimensional left U-modules V_1, V_2. The natural (U,U)-bimodule structure on U^* and on all C^V's is denoted by $x \cdot a$ and $a \cdot x$, for all $x \in U$, $a \in U^*$, where $(x \cdot a)(u) := a(ux)$ and $(a \cdot x)(u) := a(xu)$ for all $u \in U$. Note that the dual algebra U^* is a left and right U-module algebra with respect to these actions, since for all $a, b \in U^*$ and $x \in U$,

$$x \cdot (ab) = \sum (x_1 \cdot a)(x_2 \cdot b), \quad (ab) \cdot x = \sum (a \cdot x_1)(b \cdot x_2).$$

In the following we assume that U is a Hopf algebra. A **tensor category** $\mathcal{C}$ of finite dimensional left U-modules is a class $\mathcal{C}$ of finite dimensional left U-modules such that

$k \in \mathcal{C}$ (k as the trivial U-module via ε),

if $X, Y \in \mathcal{C}$, then $X \oplus Y \in \mathcal{C}$ and $X \otimes Y \in \mathcal{C}$ (with diagonal U-action on $X \otimes Y$, $u(x \otimes y) := \sum u_1 x \otimes u_2 y$ for all $u \in U$, $x \in X$ and $y \in Y$), and $X^* \in \mathcal{C}$ (where $(uf)(v) := f(S(u)v)$ for all $u \in U$, $f \in X^*$, and $v \in X$, where S denotes the antipode of U).

By definition, the dual Hopf algebra with respect to $\mathcal{C}$, $U^0_{\mathcal{C}} \subset U^0 \subset U^*$, is spanned by all matrix coefficients of all $V \in \mathcal{C}$. Thus

$$U^0_{\mathcal{C}} := \sum_{V \in \mathcal{C}} C^V.$$

A tensor category $\mathcal{C}$ is called **semisimple** if all $V \in \mathcal{C}$ are isomorphic to direct sums of simple modules in $\mathcal{C}$. If $\mathcal{C}$ is semisimple, then $U^0_{\mathcal{C}}$ is cosemisimple, hence the antipode of $U^0_{\mathcal{C}}$ is bijective. In general, if the antipode of U is bijective and for all $X \in \mathcal{C}$ also X^* with U-action given by S^{-1} (that is $(u \cdot f)(v) := f(S^{-1}(u)v)$) is in $\mathcal{C}$, then the antipode of $U^0_{\mathcal{C}}$ is bijective.

Remark 2.1: Let V be a finite dimensional left U-module with representation ρ: $U \to \mathrm{End}(V)$.

1. V is semisimple as a U-module if and only if C^V is a cosemisimple coalgebra. If V is simple, then C^V is a simple coalgebra.
2. $V \otimes V^* \to C^V$, $v \otimes f \mapsto c_{f,v}$ is a map of (U,U)-bimodules. Here, $V \otimes V^*$ is a (U,U)-bimodule via $u(v \otimes f) = uv \otimes f$ and $(v \otimes f)u := v \otimes fu$, where $(fu)(u') := f(uu')$ for all $u, u' \in U$, $v \in V$, and $f \in V^*$.

3. If V is simple and k is algebraically closed, then $V \otimes V^* \to C^V$, $v \otimes f \mapsto c_{f,v}$ is bijective.
4. Assume $\mathcal{C}$ is semisimple. Let $\mathcal{E}$ be a complete set of representatives of the simple modules in $\mathcal{C}$. Then (Peter–Weyl decomposition)

$$U^0_{\mathcal{C}} = \bigoplus_{V \in \mathcal{E}} C^V.$$

Proof: 1. If V is a semisimple U-module, then V is a faithful and semisimple $\rho(U)$-module, hence $\rho(U)$ is semisimple (since the radical of $\rho(U)$ annihilates V). If V is simple, the finite dimensional semisimple algebra $\rho(U)$ is simple (this follows, for instance, from the theorem of Artin–Wedderburn). Conversely, if $\rho(U)$ is semisimple, then V is semisimple over $\rho(U)$ and U. This proves the claim by duality since $C^V \cong \rho(U)^*$.

2. is clear.
3. By the density theorem, ρ is onto.
4. [T2, 1.4]. ∎

Let $\mathcal{C}$ be a tensor category. A subalgebra $K \subset U$ is called **$\mathcal{C}$-semisimple** if all $V \in \mathcal{C}$ are semisimple as left K-modules (by restriction).

THEOREM 2.2: *Let U be a Hopf algebra, $K \subset U$ a left coideal subalgebra and $\mathcal{C}$ a tensor category of finite dimensional left U-modules. Let $A := U^0_{\mathcal{C}}$ be the dual Hopf algebra with respect to $\mathcal{C}$, $B := \{a \in A \mid a \cdot K^+ = 0\}$ and $\bar{A} := A/AB^+$. Assume that the antipode of A is bijective. Then*

(1) *$B \subset A$ is a right coideal subalgebra with $B = {}^{\mathrm{co}\bar{A}}A$.*

(2) *If K is $\mathcal{C}$-semisimple, then $\bar{A}$ is cosemisimple and A is faithfully flat as a left and right B-module. More precisely, according to 1.5, $A = \bigoplus_{j \in J} {}_jA$ respectively $A = \bigoplus_{j \in J} S(A_j)$ is a direct sum of finitely generated and projective right respectively left B-modules and of right coideals with $B = {}_1A = S(A_1)$.*

(3) *If K is a Hopf subalgebra of U, then in (2) for all j, ${}_jA$ is also finitely generated and projective as a left B-module.*

(4) *If K is commutative and k is algebraically closed, then $\bar{A}$ is spanned by group-like elements if and only if K is $\mathcal{C}$-semisimple.*

Proof: (1) Let $\tilde{A}$ be the image of A under the restriction map $U^0 \to K^0$ which is the coalgebra map dual to the inclusion of algebras $K \subset U$. Then $\pi\colon A \to \tilde{A}$, $\pi(a)(u) := a(u)$ for all $a \in A$ and $u \in U$, is a surjective coalgebra map. Moreover, the kernel of π is a left ideal in A since for all $a \in \mathrm{Ker}(\pi)$, $c \in A$ and $u \in K$,

$(ca)(u) = \sum c(u_1)a(u_2) = 0$. Here, the last equality holds because K is a left coideal in U. Thus π: $A \to \tilde{A}$ is a surjective map of coalgebras and left A-modules. We first note that $B = {}^{\mathrm{co}\tilde{A}}A$. For if $a \in A$, then a is left $\tilde{A}$-coinvariant if and only if $\sum a_1(x)a_2(y) = \varepsilon(x)a(y)$ for all $x \in K$, $y \in U$. By definition of the right U-module structure on A (as a submodule of U^*), the last equation is equivalent to $a \cdot K^+ = 0$ or $a \in B$. Since π is a map of left A-modules, $B = {}^{\mathrm{co}\tilde{A}}A$ is a right coideal subalgebra of A, and therefore $B \subset {}^{\mathrm{co}\bar{A}}A$. To see that $B = {}^{\mathrm{co}\bar{A}}A$, note that $AB^+ \subset \mathrm{Ker}(\pi)$ since π is left A-linear and for all $b \in B^+$, $\sum \pi(b_1) \otimes b_2 = \pi(1) \otimes b$, hence $\pi(b) = \pi(1)\varepsilon(b) = 0$. Thus π can be factorized as $A \to A/AB^+ = \bar{A} \to \tilde{A}$, and ${}^{\mathrm{co}\bar{A}}A \subset {}^{\mathrm{co}\tilde{A}}A = B$.

(2) We now assume that K is $\mathcal{C}$-semisimple. Then $\tilde{A}$ is cosemisimple. For by definition, $\tilde{A}$ is the sum of all C^V restricted to K, for $V \in \mathcal{C}$. Since K is $\mathcal{C}$-semisimple, any $V \in \mathcal{C}$ is K-isomorphic to some direct sum $X_1 \oplus \cdots \oplus X_n$ of simple K-modules X_i. Hence $\pi(C^V)$ is cosemisimple as the sum of the simple subcoalgebras C^{X_i} of K^0. Thus $\tilde{A} \subset K^0$ is a cosemisimple subcoalgebra. Since $\tilde{A}$ is cosemisimple and π: $A \to \tilde{A}$ is a surjective map of coalgebras and left and right A-modules, we conclude from 1.5 that $\bar{A} = A/AB^+ = \tilde{A}$ and A is left and right faithfully flat over B and we have the decompositions of 1.5.

(3) Assume K is a Hopf subalgebra of U. Then $U^0 \to K^0$ is a Hopf algebra map and $\bar{A} = A/AB^+$ is a quotient Hopf algebra of A. The antipode defines a bijection in the set of all simple subcoalgebras C_j of $\bar{A}$. By (2) it suffices to show that $S(C_j) = C_l$ for $j, l \in J$, implies $S({}_jA) = A_l$. Indeed, for any $a \in A$,

$$\begin{aligned} a \in {}_jA &\iff \sum \bar{a}_1 \otimes a_2 \in C_j \otimes A \\ &\iff \sum S(\bar{a}_1) \otimes S(a_2) \in S(C_j) \otimes A \\ &\iff \sum S(a)_1 \otimes \overline{S(a)}_2 \in A \otimes C_l \\ &\iff S(a) \in A_l. \end{aligned}$$

(4) Assume K is commutative. Then K^0 and $\tilde{A}$ are cocommutative. If K is $\mathcal{C}$-semisimple, by (2), $\bar{A} = \tilde{A}$ is cosemisimple hence spanned by group-like elements because k is algebraically closed. Conversely, assume that A/AB^+ is spanned by group-like elements. Since $\tilde{A}$ is a coalgebra quotient of A/AB^+, also $\tilde{A}$ is spanned by group-like elements and hence cosemisimple. By definition, $\tilde{A}$ is the sum of all the dual coalgebras $\rho_V(K)$, where ρ_V: $U \to \mathrm{End}(V)$ is the representation corresponding to $V \in \mathcal{C}$. Hence for all $V \in \mathcal{C}$, $\rho_V(K)^*$ is cosemisimple or equivalently $\rho_V(K)$ is a semisimple algebra or V is a semisimple module over K. Thus K is $\mathcal{C}$-semisimple. ∎

Remark 2.3: In the proof of 2.2 we showed that A is left and right faithfully flat over B if A is left faithfully coflat over $\tilde{A}$ and that this last condition is satisfied in case K is $\mathcal{C}$-semisimple. More generally let $\mathcal{F}$ be the set of all U-annihilators of all $V \in \mathcal{C}$ and assume that for all $I \in \mathcal{F}$ there is $J \in \mathcal{F}$ with $J \subset I$ such that U/J is left faithfully flat over $K/K \cap J$. Then A is left faithfully coflat over $\tilde{A}$.

In the situation of 2.2(4) we now assume that also $\mathcal{C}$ is semisimple. Then the decompositions in 2.2 can be described more concretely. In this case we will give an alternative proof not using 2.2. We will explicitly determine dual bases of the finitely generated and projective B-module summands of A and the set $G(\bar{A})$ of group-like elements of the quotient coalgebra $\bar{A}$.

For any character $\chi \in \mathrm{Alg}(K,k)$ of K, a left K-module M, and a right K-module N we will denote the eigenspaces of χ by

$$_{\chi}M := \{m \in M \mid \forall x \in K\colon xm = \chi(x)m\}$$

and

$$N_{\chi} := \{n \in N \mid \forall x \in K\colon nx = n\chi(x)\}.$$

COROLLARY 2.4: *Let U be a Hopf algebra with bijective antipode over an algebraically closed field k, $K \subset U$ a left coideal subalgebra and $\mathcal{C}$ a tensor category of finite dimensional left U-modules. Define A, B and $\bar{A}$ as in 2.2. Assume that K is commutative and $\mathcal{C}$-semisimple and $\mathcal{C}$ is semisimple. Let $X(K,\mathcal{C})$ be the set of all $\chi \in \mathrm{Alg}(K,k)$ with $_{\chi}V \neq \{0\}$ for some simple U-module $V \in \mathcal{C}$. Then*

1. *$A = \bigoplus_{\chi \in X(K,\mathcal{C})} A_{\chi}$ and $A = \bigoplus_{\chi \in X(K,\mathcal{C})} S(_{\chi}A)$ are direct sums of right coideals and finitely generated and projective right (left, respectively) B-modules with $B = A_{\varepsilon} = S(_{\varepsilon}A)$.*
2. *The natural coalgebra map $\bar{A} \to K^0$ given by restriction defines a bijection*

$$G(\bar{A}) \xrightarrow{\cong} X(K,\mathcal{C}), \quad g \mapsto \chi_g,$$

and for any group-like element $g \in \bar{A}$, the right eigenspace A_{χ_g} of χ_g is the space of left g-invariant elements $_gA := \{a \in A \mid \sum \bar{a}_1 \otimes a_2 = g \otimes a\}$, and $_{\chi_g}A = A_g := \{a \in A \mid \sum a_1 \otimes \bar{a}_2 = a \otimes g\}$.

Proof: (1) We first consider the decomposition $A = \bigoplus_{\chi} A_{\chi}$. Let $\mathcal{E}$ be a set of representatives of the isomorphism classes of the simple modules in $\mathcal{C}$. By 2.1, $A = \bigoplus_{V \in \mathcal{E}} C^V$ is a decomposition into finite dimensional right (and left) U-modules. Since K is commutative and k is algebraically closed, all finite dimensional simple K-modules are 1-dimensional and given by characters of K.

By assumption any $V \in \mathcal{E}$ is K-semisimple and has a basis (v_i) of eigenvectors $v_i \in V_{\chi_i}$ for some characters χ_i of K. Hence $xv_i = \chi_i(x)v_i$ for all i and all $x \in K$. Let (f_i) be the dual basis of (v_i) in V^*. Then $f_i \cdot x = \chi_i(x)f_i$ for all i and all $x \in K$. Thus it follows from 2.1, (2) and (3) that for all $V \in \mathcal{E}$, $C^V \cong V \otimes V^*$ is a semisimple right K-module, and we can write

$$A = \bigoplus_{\chi} A_\chi := \bigoplus_{V \in \mathcal{E}} \bigoplus_{\chi} (C^V)_\chi .$$

By definition, $A_\varepsilon = B$. For any χ, A_χ is a right B-submodule of A since for all $a \in A_\chi$, $b \in B$ and $x \in K$,

$$\begin{aligned}(ab) \cdot x =& \sum (a \cdot x_1)(b \cdot x_2)\\ =& \sum (a \cdot x_1) b\varepsilon(x_2), \text{ since } x_2 \in K\\ =&(a \cdot x)b\\ =&ab\chi(x).\end{aligned}$$

It is easy to check that all eigenspaces A_χ are right coideals in A.

Let χ be a character of K. It remains to show that A_χ is finitely generated and projective as a right B-module. We can assume that $A_\chi \neq \{0\}$ or equivalently $\chi \in X(K,C)$. Hence there exists a simple module $V \in \mathcal{E}$ with $(C^V)_\chi \neq \{0\}$. Let (v_i), (f_i) be dual bases of eigenvectors of V and V^* as before. Thus $f_j \cdot x = \chi(x)f_j$ and $xv_j = \chi(x)v_j$ for some j and all $x \in K$. Define $v := v_j$, $f := f_j$ and consider the matrix coefficient $c_{f,v}$. Since (v_i), (f_i) are dual bases and

$$\Delta(c_{f,v}) = \sum_i c_{f,v_i} \otimes c_{f_i,v},$$

we get $1 = f(v) = \varepsilon(c_{f,v}) = \sum_i c_{f,v_i} S(c_{f_i,v})$, where S is the antipode of A. Define $a_i := c_{f,v_i}$, $b_i := S(c_{f_i,v})$ and $\phi_i(a) := b_i a$ for all $a \in A_\chi$. By the dual basis lemma it suffices to prove that $a_i \in A_\chi$ and $\phi_i(a) = b_i a \in B$ for all i and $a \in A_\chi$. For all $u \in U$ and $x \in K$,

$$(a_i \cdot x)u = f(xuv_i) = (f \cdot x)(uv_i) = \chi(x)f(uv_i) = \chi(x)a_i(u),$$

hence $a_i \in A_\chi$.

Finally, for all $a \in A_\chi$ and $x \in K$,

$$\begin{aligned}(b_i a) \cdot x =& \sum (b_i \cdot x_1)(a \cdot x_2)\\ =& \sum (b_i \cdot x_1) a\chi(x_2) \quad \text{since } x_2 \in K\\ =&(b_i \cdot \sum x_1 \chi(x_2))a\\ =&(b_i a)\varepsilon(x),\end{aligned}$$

since $(b_i \cdot \sum x_1\chi(x_2)) = b_i\varepsilon(x)$. Thus $b_i \cdot a \in B$. The last equality holds since for all $u \in U$,

$$\begin{aligned}(b_i \cdot \sum x_1\chi(x_2))(u) =& S(c_{f_i,v})(\sum x_1\chi(x_2)u) \\ =& c_{f_i,v}(S(u)\sum S(x_1)\chi(x_2)) \\ =& f_i(S(u)\sum S(x_1)\chi(x_2)v) \\ =& f_i(S(u)\sum S(x_1)x_2v) \qquad \text{since } x_2 \in K \\ =& f_i(S(u)v)\varepsilon(x) = b_i(u)\varepsilon(x).\end{aligned}$$

In the same way we have a decomposition into left eigenspaces $A = \bigoplus_\chi {}_\chi A$. Hence $A = \bigoplus_\chi S({}_\chi A)$ since S is bijective. One easily sees that all the $S({}_\chi A)$ are right coideals. From Koppinen's lemma for left coideals (apply 1.4 to the dual coalgebra) we get

$$\begin{aligned}K^+U = S^{-1}(UK^+) = S^{-1}(K^+)U \quad \text{and} \\ B = \{a \in A \mid a \cdot S^{-1}(K^+) = \{0\}\}.\end{aligned}$$

Hence $S({}_\varepsilon A) = B$ since for all $a \in A$, $x \in K$ and $u \in U$:

$$\left(S(a) \cdot S^{-1}(x)\right)(u) = (x \cdot a)(S(u)).$$

Then we see that $S({}_\chi A)$ is a left B-submodule of A for any χ since for all $a \in {}_\chi A$, $b \in B$ and $x \in K$,

$$\begin{aligned}x \cdot S^{-1}(bS(a)) =& x \cdot (aS^{-1}(b)) = \sum (x_1 \cdot a)(x_2 \cdot S^{-1}(b)) \\ =& \sum (x_1 \cdot a)\varepsilon(x_2)S^{-1}(b) \qquad \text{since } S^{-1}(B) = {}_\varepsilon A \\ =& (x \cdot a)S^{-1}(b) \\ =& \varepsilon(x)S^{-1}(bS(a)).\end{aligned}$$

Let $\chi \in X(K, C)$. We have to show that $S({}_\chi A)$ is finitely generated and projective as a left B-module. Using the notations above it suffices to show that $b_i \in S({}_\chi A)$ and $S(a)a_i \in B$ for all i and $a \in {}_\chi A$. For all $u \in U$ and $x \in K$,

$$(x \cdot c_{f_i,v})(u) = f_i(uxv) = \chi(x)f_i(uv) = \chi(x)c_{f_i,v}(u),$$

hence $b_i = S(c_{f_i,v}) \in S({}_\chi A)$. For all $a \in {}_\chi A$, $S(a)a_i \in B = S({}_\varepsilon A)$ since $a_i = c_{f,v_i}$

and for all $x \in K$,

$$\begin{aligned}x \cdot S^{-1}(S(a)a_i) =& x \cdot (S^{-1}(c_{f,v_i})a)\\ =& \sum (x_1 \cdot S^{-1}(c_{f,v_i}))(x_2 \cdot a)\\ =& \sum (x_1 \cdot S^{-1}(c_{f,v_i}))\chi(x_2)a \quad \text{since } a \in {}_\varepsilon A\\ =& ((\sum \chi(x_2)x_1) \cdot S^{-1}(c_{f,v_i}))a\\ =& \varepsilon(x)S^{-1}(c_{f,v_i})a\\ =& \varepsilon(x)S^{-1}(S(a)a_i).\end{aligned}$$

In the proof we used the equality

$$\sum \chi(x_2)x_1 \cdot S^{-1}(c_{f,v_i}) = \varepsilon(x)S^{-1}(c_{f,v_i}),$$

which holds since for all $u \in U$

$$\begin{aligned}(\sum \chi(x_2)x_1 \cdot S^{-1}(c_{f,v_i}))(u) =& \sum c_{f,v_i}(\chi(x_2)S^{-1}(x_1)S^{-1}(u))\\ =& \sum f(\chi(x_2)S^{-1}(x_1)S^{-1}(u)v_i)\\ =& \sum f(x_2S^{-1}(x_1)S^{-1}(u)v_i) \quad \text{since } f \in (V^*)_\chi\\ =& \varepsilon(x)f(S^{-1}(u)v_i) = \varepsilon(x)S^{-1}(c_{f,v_i})(u).\end{aligned}$$

(2) As in the proof of 2.2, let $\tilde{A}$ be the image of A under the restriction map $U^0 \to K^0$ and $\pi: A \to \tilde{A}$ the induced surjective coalgebra map. By 2.2, the kernel of π is AB^+. The group-like elements in A^0 are the characters of K. Hence the group-like elements $G(\tilde{A})$ of $\tilde{A}$ are the characters of K which can be extended to a linear map $a: U \to k$ with $a \in A$.

Let χ be a character of K with $A_\chi \neq 0$. In the notation of the proof of (1), the matrix coefficient $c_{f,v}$ is an element in A such that for all $x \in K$, $c_{f,v}(x) = \chi(x)$. Therefore χ is a group-like element of $\tilde{A}$.

By definition, the space of left χ-invariants is the eigenspace A_χ, since for all $a \in A$, $\sum \pi(a_1) \otimes a_2 = \chi \otimes a$ if and only if for all $x \in K$ and $u \in U$, $a(xu) = \sum a_1(x)a_2(u) = \chi(x)a(u)$. Similarly, the space of right χ-invariants is ${}_\chi A$. In particular for all $a \in A_\chi$, $\pi(a) = \chi\varepsilon(a)$, and we see that $\pi(A_\chi) = k\chi$.

Hence by (1), $\tilde{A} = \pi(A) = \bigoplus_{\chi \in X(K,\mathcal{C})} k\chi$, and $X(K,\mathcal{C})$ is the set of all group-like elements of $\tilde{A}$. This finishes the proof of (2) since $\bar{A} \cong \tilde{A}$. ∎

In the next theorem we consider the special case of 2.2 when $K = k[x]$ for some $(g,1)$-**primitive element** $x \in U$, that is $\Delta(x) = g \otimes x + x \otimes 1$, g group-like in U. Then K is a commutative left coideal subalgebra of U. If $\mathcal{C}$ is a tensor category of finite dimensional left U-modules, we call an element $x \in U$ **$\mathcal{C}$-semisimple** if for all $V \in \mathcal{C}$, the linear map $V \to V$, $v \mapsto xv$, is diagonalizable. When x is $(1, g)$-primitive, that is $\Delta(x) = 1 \otimes x + x \otimes g$, then $k[x]$ is a right coideal subalgebra of U.

THEOREM 2.5: *Let U be a Hopf algebra with bijective antipode over an algebraically closed field k, g a group-like element and x a $(g, 1)$-primitive (resp. $(1, g)$-primitive) element of U and $\mathcal{C}$ a tensor category of finite dimensional left U-modules. Define $A := U^0_{\mathcal{C}}$, $B := \{a \in A \mid a \cdot x = 0\}$ and $\bar{A} := A/AB^+$ (resp. A/B^+A). Assume that the antipode of A is bijective.*

(1) *The following are equivalent:*
  (a) *x is $\mathcal{C}$-semisimple.*
  (b) *$\bar{A}$ is spanned by group-like elements.*
  If these conditions hold, then A is faithfully flat as a right and as a left B-module .

(2) *If $\mathcal{C}$ is semisimple and B is a simple object in $\mathcal{M}^A_B$ (resp. ${}_B\mathcal{M}^A$), then $B := \{a \in A \mid \exists n \geq 1\colon a \cdot x^n = 0\}$.*

Proof: We assume that x is $(g, 1)$-primitive (in the case of $(1, g)$-primitive elements we then apply the result to the dual coalgebra of U). Then $K := k[x]$ is a left coideal subalgebra of U and (1) is a special case of 2.2(4) and (2) since x is $\mathcal{C}$-semisimple if and only if K is $\mathcal{C}$-semisimple.

For (2) it is enough to show that $B = A^{(2)} := \{a \in A \mid a \cdot x^2 = 0\}$. Assume $B \subsetneq A^{(2)}$. Let $\phi\colon A \to A$, $\phi(a) := a \cdot x$, be right multiplication with x. The map ϕ is right A-colinear since for all $a \in A$ and $u, u' \in U$,

$$\sum (a_1 \cdot x)(u) a_2(u') = \sum a_1(xu) a_2(u') = a(xuu') = (a \cdot x)(uu'),$$

hence $\Delta(a \cdot x) = \sum a_1 \cdot x \otimes a_2$. Since A is a right U-module algebra, ϕ is also right B-linear, because for all $a \in A$ and $b \in B$,

$$(ab) \cdot x = \sum (a \cdot x_1)(b \cdot x_2) = (a \cdot g)(b \cdot x) + (a \cdot x)b = (a \cdot x)b$$

since $b \cdot x = 0$ by the definition of B. Thus ϕ is a morphism in $\mathcal{M}^A_B$. Hence also $A^{(2)} \to B$, $a \mapsto a \cdot x$, is a morphism in $\mathcal{M}^A_B$. This map is non-zero since $B \subsetneq A^{(2)}$, hence it is onto because B is a simple object in $\mathcal{M}^A_B$. By assumption, $\mathcal{C}$ is semisimple, hence A is cosemisimple. Therefore the surjective map $A^{(2)} \to B$ splits as a map of right A-comodules and there exists a right A-colinear map $\gamma\colon B \to A^{(2)}$ such that $\gamma(b) \cdot x = b$ for all $b \in B$. Since $\Delta(\gamma(1)) = \gamma(1) \otimes 1$, $\gamma(1)$ is a scalar multiple of the identity in B and we get the contradiction $1 = \gamma(1) \cdot x = \gamma(1)\varepsilon(x) = 0$. ∎

3. Unitarizable tensor categories

In this short section we want to show that for Hopf $*$-algebras there is a natural condition which guarantees that a left coideal subalgebra K is $\mathcal{C}$-semisimple.

Hence in many examples for quantum homogeneous spaces with $*$-structures our main result 2.2 can be applied.

A Hopf $*$-algebra H [KS, 1.2.7] is a Hopf algebra over the field of complex numbers with an involution $*$: $H \to H$ such that for $x, y \in H$ and a complex number α (with complex conjugate $\bar{\alpha}$)

$$(x+y)^* = x^* + y^*,\ (\alpha x)^* = \bar{\alpha} x^*,\ (xy)^* = y^* x^*,\ \Delta(x^*) = \sum x_1^* \otimes x_2^*.$$

Then $1^* = 1$, $\varepsilon(x^*) = \overline{\varepsilon(x)}$ and $S(x^*)^* = S^{-1}(x)$ for all $x \in H$.

In this section, for all subsets T of a Hopf $*$-algebra H let $T^* := \{t^* \mid t \in T\}$.

A coalgebra is called **pointed** if each of its simple subcoalgebras is one-dimensional. A Hopf algebra is pointed if it is generated as an algebra by group-like and skew-primitive elements [M, 5.5.1]. In particular, the q-deformed enveloping algebras of semisimple Lie algebras are all pointed.

PROPOSITION 3.1: *Let U be a Hopf $*$-algebra and $I \subset U$ a coideal. Define $K := U^{\mathrm{co}\, U/IU}$. Then:*

(1) *K is a left coideal subalgebra of U, and if U is pointed (or more generally U is faithfully left coflat over U/IU), then $IU = K^+U$.*
(2) *If $S(I)^* \subset I$, then $K^* = K$.*

Proof: (1) Since IU is a right ideal and coideal, the set of right U/IU-coinvariant elements K is a left coideal subalgebra. If U is pointed, then by [M2, 1.3] U is left (and right) faithfully coflat over U/IU. Hence we know from Theorem 1.1 (applied to $A^{\mathrm{op\,cop}}$) that $IU = K^+U$.

(2) First note that $K^+ \subset IU$ since, for any $x \in K$, $\sum x_1 \otimes \bar{x}_2 = x \otimes \bar{1}$, hence $\bar{x} = \varepsilon(x)\bar{1}$ in U/IU. By Koppinen's lemma 1.4, $S(K^+U) = UK^+$. Then

$$\begin{aligned}(K^+)^* \subset (UK^+)^* &= S(K^+U)^*, \quad \text{since } UK^+ = S(K^+U)\\ &\subset S(IU)^*, \quad \text{since } K^+ \subset IU\\ &= S(I)^*U\\ &\subset IU, \quad \text{since } S(I)^* \subset I.\end{aligned}$$

Clearly K^* is again a left coideal subalgebra with $(K^*)^+ = (K^+)^*$, and we have shown that $(K^*)^+U \subset IU$. Therefore

$$K^* \subset U^{\mathrm{co}\, U/(K^*)^+U} \subset U^{\mathrm{co}\, U/IU} = K.$$

Since $*$ is an involution, we get $K^* = K$. ∎

As an illustration of 3.1(1), let g, h be group-like elements of U and $x \in U$ with $\Delta(x) = g \otimes x + x \otimes h$. Then kx is a one-dimensional coideal, $k[xh^{-1}]$ is a left coideal subalgebra, and $xU = k[xh^{-1}]^+U$.

Remark 3.2: Let U be a Hopf $*$-algebra and $\mathcal{C}$ a tensor category of finite dimensional left U-modules. For any left U-module V let $\bar{V}$ be V as an additive group with the following left U-module structure, denoted by $\star$: For any $v \in V$ and $u \in U$,

$$u \star v := S(u)^* v.$$

In particular, $\bar{V}$ is a complex vector space with $\alpha \star v := \bar{\alpha} v$ for complex numbers α. $\bar{V}$ is the restriction of V to U via the ring isomorphism

$$U \to U, \quad u \mapsto S(u)^*.$$

A tensor category $\mathcal{C}$ will be called a **tensor $*$-category** if for all $V \in \mathcal{C}$ also $\bar{V} \in \mathcal{C}$. Let $\mathcal{C}$ be a tensor $*$-category. Then:

(1) $A := U^0_{\mathcal{C}}$ is a Hopf $*$-algebra with $*$-structure defined by $a^*(u) := \overline{a(S(u)^*)}$ for all $a \in A$ and $u \in U$.

(2) If $I \subset U$ is a coideal with $S(I)^* \subset I$, then $B := \{a \in A \mid a \cdot I = 0\}$, the algebra of infinitesimal invariants defined by I, is a $*$-subalgebra of A.

Proof: (1) The full Hopf dual U^0 is a Hopf $*$-algebra with $*$-structure as described above. A is closed under the $*$-structure since for all $V \in \mathcal{C}$, $v \in V$, linear functionals f on V and $u \in U$,

$$c^*_{f,v}(u) = \overline{f(S(u)^* v)} = c_{\bar{f},v}(u),$$

where $c_{\bar{f},v}$ is the matrix coefficient of $\bar{V}$ and the linear functional $\bar{f}$ on $\bar{V}$ is defined by $\bar{f}(w) := \overline{f(w)}$ for all $w \in \bar{V} = V$.

(2) is easy to check [KD, 1.9]. ∎

Let U be a Hopf $*$-algebra and $\mathcal{C}$ a tensor category of finite dimensional left U-modules. We call $\mathcal{C}$ **unitarizable** if for all $V \in \mathcal{C}$ there is a hermitian inner product $\langle , \rangle$: $V \times V \to \mathbb{C}$, conjugate linear in the first and linear in the second variable such that for all $x \in U$ and $v, w \in V$, $\langle xv, w \rangle = \langle v, x^* w \rangle$.

COROLLARY 3.3: *Let U be a pointed Hopf $*$-algebra, $\mathcal{C}$ a unitarizable tensor category of finite dimensional left U-modules, and $I \subset U$ a coideal with $S(I)^* \subset I$. Define $A := U^0_{\mathcal{C}}$ and $B := \{a \in A |\ a \cdot I = 0\}$. Then:*

(1) *$K := U^{\mathrm{co}\, U/IU}$ is a left coideal subalgebra of U, $B := \{a \in A \mid \forall x \in K\colon a \cdot x = a\varepsilon(x)\}$ is a right coideal subalgebra of A and K is $\mathcal{C}$-semisimple.*

(2) *A/AB^+ is cosemisimple, A is faithfully flat as a left and a right B-module and a direct sum of finitely generated and projective left B-modules and right B-modules as in 2.2.*

Proof: (1) Since U is pointed, we get from 3.1(1) that $IU = K^+U$. Hence B is also the set of infinitesimal invariants with respect to K^+. Moreover, $K^* = K$ by 3.1(2). Let $V \in \mathcal{C}$ and $\langle , \rangle$ the hermitian inner product on $V \times V$. We have to show that V is semisimple as a module over K. Let $W \in V$ be a K-submodule. Then $W^\perp = \{v \in V \mid \forall w \in W\colon \langle v, w\rangle = 0\}$ is a K-submodule because for all $x \in K$, $v \in W^\perp$, and $w \in W$,

$$\langle xv, w\rangle = \langle v, x^*w\rangle = 0$$

since $x^* \in K$. Hence $V = W \oplus W^\perp$ is a direct sum of K-modules, and V is K-semisimple.

(2) follows from (1) and 2.2. ∎

Remark 3.4: Corollary 3.3 applies to many recent examples of quantum homogeneous spaces.

1. In general, if $\mathfrak{g}$ is a semisimple Lie algebra and $U = U_q(\mathfrak{g})$ is the q-deformed universal enveloping algebra with positive real $q \neq 1$, then U is a pointed Hopf $*$-algebra with standard $*$-structure, and the tensor category of finite dimensional left U-modules of type 1 is unitarizable ([CP, 10.1.21], for the non-degeneracy of the inner product on the simple modules cf. [dCK, Proposition 1.9]).

2. Let $A = A(\mathrm{SU}_q(2))$ be the function algebra of the q-deformed special unitary group SU(2) and assume $0 < q \leq 1$. For any parameter $\rho \in [0, \infty]$, Dijkhuizen and Koornwinder [KD], [KS, 4.5] defined a right coideal subalgebra $B = B_\rho \subset A$ by infinitesimal invariants with respect to one skew-primitive element x_ρ. They show that the algebras B_ρ can be identified with the function algebras of Podleś' quantum spheres S^2_{qc} for $0 \leq c \leq \infty$, [P]. In this situation all the assumptions in 3.3 are satisfied for the extension $B \subset A$. In particular, we know from 3.3 and 2.2(4) that A/AB^+ is spanned by group-like elements. This answers a question of Brzeziński [B].

3. A more general family of examples has been introduced by Dijkhuizen and Noumi [DN], [KS, 11.6.6]. Let $n \geq 2$ and $0 < q \leq 1$. For any non-negative real numbers c, d which do not vanish simultaneously, they define a coideal $I = I^{(c,d)}$ in $U_q(\mathfrak{gl}(n))$. Then they define the quantum projective space $\mathbb{CP}_q^{n-1}(c,d)$ with a natural transitive action of the quantum unitary group $U_q(n)$. Here $\mathbb{CP}_q^{n-1}(c,d)$ is given by a right coideal subalgebra $B = B^{(c,d)}$ in $A := A(U_q(n))$, the function algebra of $U_q(n)$. B is defined by infinitesimal invariants with respect to $I^{(c,d)}$. These homogeneous $U_q(n)$-spaces have first been studied by Vaksman and Korogodsky [KV] as a q-analog of the Hopf fibration $S^{2n-1} \to \mathbb{CP}^{n-1}$. Again the extension $B \subset A$ satisfies the assumptions in 3.3.

4. [D] contains a survey of similar constructions of quantum homogeneous spaces which are analogs of compact symmetric spaces.

4. Semisimple skew-primitive elements in $U_q(\mathrm{sl}(2))$

Let k be an algebraically closed field and let q be a non-zero element of k, which is not a root of unity. For all positive integers n let E_n be the unit matrix with n rows and columns.

First we recall some definitions and results of [K].

For any integer n, set [K, p. 121]

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{-n+3} + q^{-n+1}.$$

The Hopf algebra $U_q(\mathrm{sl}(2))$ is generated as an algebra by E, F, K, K^{-1} with relations

$$KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = q^2E,$$
$$KFK^{-1} = q^{-2}F, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

and comultiplication defined by

$$\Delta(K) = K \otimes K, \quad \Delta(K^{-1}) = K^{-1} \otimes K^{-1},$$
$$\Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1.$$

All simple $U_q(\mathrm{sl}(2))$ left modules are isomorphic to some $V_{e,n}$, where $e = \pm 1$ and n is a nonnegative integer. The $(n+1)$-dimensional module $V_{e,n}$ has a basis $\{v_0, v_1, \ldots, v_n\}$, such that the left action of the generators E, F, and K of

$U_q(\mathrm{sl}(2))$ can be represented on this basis by the matrices [K, p. 129]

$$\rho_{e,n}(E)=e\begin{pmatrix}0&[n]&0&\cdots&0\\0&0&[n-1]&\cdots&0\\\vdots&\ddots&\ddots&\ddots&\vdots\\0&0&\ddots&\ddots&1\\0&0&\cdots&0&0\end{pmatrix},\quad \rho_{e,n}(F)=\begin{pmatrix}0&0&\cdots&0&0\\1&0&\cdots&0&0\\0&[2]&\ddots&0&0\\\vdots&\ddots&\ddots&\ddots&\vdots\\0&0&\cdots&[n]&0\end{pmatrix},$$

$$\text{and}\quad \rho_{e,n}(K)=e\begin{pmatrix}q^n&0&\cdots&0&0\\0&q^{n-2}&\cdots&0&0\\\vdots&\ddots&\ddots&\vdots&\vdots\\0&0&\cdots&q^{-n+2}&0\\0&0&\cdots&0&q^{-n}\end{pmatrix}.$$

Here we need the right action and use the transposed matrices. We first consider the case $e=1$. Let

$$x=\alpha(K^{-1}-1)+\beta EK^{-1}+\gamma F$$

be a $(K^{-1},1)$-primitive element of $U_q(\mathrm{sl}(2))$, where α, β, and γ are fixed elements of the ground field k, which do not vanish simultaneously. Then the (right) action of x on the chosen basis of $V_{1,n}$ can be represented by the matrix

$$M_n:=\begin{pmatrix}(q^{-n}-1)\alpha&\gamma&0&\cdots&0\\q^{-n}[n]\beta&(q^{2-n}-1)\alpha&[2]\gamma&\ddots&\vdots\\0&q^{2-n}[n-1]\beta&(q^{4-n}-1)\alpha&\ddots&0\\\vdots&\ddots&\ddots&\ddots&[n]\gamma\\0&\cdots&0&q^{n-2}[1]\beta&(q^n-1)\alpha\end{pmatrix}.$$

The **quantum plane** $k_q[a,b]$ is the k-algebra generated by the elements a and b with the relation $ba=qab$ [K, Chapter IV]. It is a $U_q(\mathrm{sl}(2))$ left module algebra, where the action of $U_q(\mathrm{sl}(2))$ is given in [K, Chapter VII.3]. Here we need the corresponding right action given by

$$b\cdot E=0,\ a\cdot E=b,\ b\cdot F=a,\ a\cdot F=0,\ b\cdot K=qb,$$
$$a\cdot K=q^{-1}a,\ b\cdot K^{-1}=q^{-1}b,\ a\cdot K^{-1}=qa.$$

The quantum plane has a natural gradation, given by the degrees of the monomials, therefore there are no zero divisors. Let $k_q[a,b]_n$ denote the vector subspace of homogeneous polynomials of degree n in $k_q[a,b]$. Then $k_q[a,b]_n$ is a simple $U_q(\mathrm{sl}(2))$ right module isomorphic to $V_{1,n}$ [K, Theorem VII.3.3].

LEMMA 4.1:

(1) *There is a non-zero element* $\xi \in k_q[a,b]_2$ *such that* $\xi \cdot x = 0$.

(2) *For all* $z, \xi \in k_q[a,b]$, *where* $\xi \cdot x = 0$, *the equation* $(z\xi)\cdot x = (z \cdot x)\xi$ *holds, that is, the action of* x *is* B_X*-right linear, where*

$$B_X = \{\xi \in k_q[a,b] \colon \xi \cdot x = 0\}.$$

(3) *The eigenspaces of the action of* x *on* $k_q[a,b]_n$ *are one-dimensional.*

Proof: (1) The determinant of M_2 vanishes.

(2) The quantum plane is a module algebra of $U_q(\mathrm{sl}(2))$:

$$(z\xi)\cdot x = (z \cdot K^{-1})(\xi \cdot x) + (z \cdot x)(\xi \cdot 1) = (z \cdot x)\xi.$$

(3) Let $\lambda \in k$ and let E_{n+1} be the unit matrix with $n+1$ rows and columns. In the cases $\beta \neq 0$ or $\gamma \neq 0$ the first or last n columns respectively of $M_n - \lambda E_{n+1}$ are linearly independent. In the case $\beta = \gamma = 0$ there must be $u \neq 0$, and in the diagonal matrix $M_n - \lambda E_{n+1}$ at most one entry in the diagonal vanishes, for since q is not a root of unity, all entries in the diagonal are pairwise distinct. ∎

For convenience, we fix a solution $\sqrt{q}$ of the equation $x^2 = q$ and define $q^l := \sqrt{q}^{2l}$ for $l \in \frac{1}{2}\mathbb{Z}$. For each nonnegative integer n set

$$I_n := \{-\tfrac{n}{2}, 1 - \tfrac{n}{2}, \ldots, \tfrac{n}{2} - 1, \tfrac{n}{2}\}.$$

PROPOSITION 4.2:

(1) *Let*

$$P_n(Y) = \det(M_n - YE_{n+1}) = (-1)^{n+1}(Y^{n+1} + z_n Y^n + d_{n-1}Y^{n-1} + \cdots)$$

be the characteristic polynomial of M_n. *Then for* $n \geq 2$ *the polynomial* P_{n-2} *divides* P_n.

(2) *Fix* $R \in k$ *such that*

$$R^2 = \alpha^2 + \frac{4\beta\gamma q^{-1}}{(q-q^{-1})^2}.$$

Then M_n *has the* $n+1$ *(not necessarily distinct) eigenvalues*

$$\lambda_r := \frac{\alpha}{2}(q^r - q^{-r})^2 + \frac{1}{2}(q^{2r} - q^{-2r})R,$$

for $r \in I_n$.

Proof: (1) Let μ be an f-fold zero of P_{n-2}, i.e. the dimension of the generalized eigenspace for the eigenvalue μ in $k_q[a,b]_{n-2}$ is f, and let $v_1, \ldots, v_f$ be a basis of V_μ^{n-2}. Therefore there exists a positive integer t such that $v_\nu \cdot (x-\mu)^t = 0$ for $\nu = 1, \ldots, f$. Let ξ be as in Lemma 4.1. Then $v_1\xi, \ldots, v_f\xi$ are linearly independent elements of $k_q[a,b]_n$, since $k_q[a,b]$ is an integral domain. Moreover, by part (2) of the lemma,

$$(v_\nu\xi)\cdot(x-\mu)^t = \left(v_\nu \cdot (x-\mu)^t\right)\xi = 0,$$

whence $v_\nu\xi$ for $\nu = 1, \ldots, f$ belongs to the generalized eigenspace for the eigenvalue μ in $k_q[a,b]_n$. Therefore μ is at least an f-fold zero of P_n.

(2) For $n = 0$ and $n = 1$ one easily computes

$$\begin{aligned}&P_0(Y) = 0 - Y = -Y \text{ and}\\ &P_1(Y) = Y^2 - \alpha(\sqrt{q} - \sqrt{q}^{-1})^2 Y - q^{-1}\beta\gamma - \alpha^2(\sqrt{q} - \sqrt{q}^{-1})^2.\end{aligned}$$

Now assume $n \geq 2$. Due to $\deg P_n(Y) = 2 + \deg P_{n-2}(Y)$ and part (a) only the two extra zeros a_n and b_n of P_n have to be determined. Vieta's Theorem says

$$\begin{aligned}z_{n-2} - a_n - b_n = z_n,\ a_n b_n - (a_n + b_n) z_{n-2} + d_{n-2} = d_n \ \Rightarrow\\ a_n + b_n = z_{n-2} - z_n,\ a_n b_n = d_n - d_{n-2} + z_{n-2}(z_{n-2} - z_n).\end{aligned}$$

Therefore a_n and b_n must satisfy the quadratic equation in T

$$T^2 - (z_{n-2} - z_n)T + d_n - d_{n-2} + z_{n-2}(z_{n-2} - z_n) = 0.$$

Let $l = n/2$. We compute the coefficients of this equation:

(a) The coefficient z_n is the negative trace of M_n. Therefore

$$z_n - z_{n-2} = -u(q^n - 1 + q^{-n} - 1) = -u(q^l - q^{-l})^2.$$

(b) The coefficient d_n is the sum of D_n and N_n, where D_n is the sum of all products of two distinct entries of the main diagonal and N_n is the sum of products of entries, which do not belong to the main diagonal. All indices run over I_n if there are no limits of summation.

$$\begin{aligned}D_n :=& \alpha^2 \sum_{r<s} (q^{-2r} - 1)(q^{-2s} - 1)\\ =& \frac{1}{2}\Big(\sum_r \alpha(q^{-2r} - 1)\Big)^2 - \frac{1}{2}\Big(\alpha^2 \sum_r (q^{-2r} - 1)^2\Big)\\ =& \frac{1}{2} z_n^2 - \frac{1}{2}\alpha^2 \sum_r (q^{-2r} - 1)^2.\end{aligned}$$

$$N_n := -q^{-2}\beta\gamma\sum_{t=1}^{n} q^{2t-n}[t][n+1-t] \Rightarrow$$

$$\begin{aligned}
\frac{-N_n q}{\beta\gamma/(q-q^{-1})^2} =& \sum_{t=1}^{n} q^{2t-n-1}(q^t - q^{-t})(q^{n+1-t} - q^{t-n-1})\\
=& (q^{n+1}+q^{-n-1})\sum_{t=1}^{n} q^{2t-n-1} - \sum_{t=1}^{n}(q^{4t-2n-2}+1)\\
=& (q^{n+1}+q^{-n-1})\frac{q^n - q^{-n}}{q-q^{-1}} - \sum_{t=1}^{n}(q^{4t-2-2n}+1)\\
=& \frac{q^{2n+1}-q^{-2n-1}}{q-q^{-1}} - 1 + \sum_{t=1}^{n}(q^{4t-2-2n}+1) \Rightarrow\\
\frac{N_{l-1}-N_l}{q^{-1}\beta\gamma/(q-q^{-1})^2} =& \frac{q^{2n+1}-q^{2n-3}+q^{3-2n}-q^{-2n-1}}{q-q^{-1}} - q^{2n-2} - q^{2-2n} - 2\\
=& (1+q^2)(q^{2n-2}+q^{-2n}) - q^{2n-2} - q^{2-2n} - 2\\
=& (q^n - q^{-n})^2.
\end{aligned}$$

Now the constant coefficient of the quadratic equation can be computed:

$$\begin{aligned}
d_l - d_{l-1} + z_{l-1}(z_{l-1}-z_l) =& N_l - N_{l-1} + D_l - D_{l-1} + z_{l-1}(z_{l-1}-z_l)\\
=& N_l - N_{l-1} - \tfrac{1}{2}\alpha^2\left((q^n-1)^2 + (q^{-n}-1)^2\right)\\
&+ \tfrac{1}{2}z_l^2 - \tfrac{1}{2}z_{l-1}^2 + z_{l-1}^2 - z_l z_{l-1}\\
=& N_l - N_{l-1} - \tfrac{1}{2}\alpha^2(q^n(q^l-q^{-l})^2 + q^{-n}(q^l-q^{-l})^2)\\
&+ \tfrac{1}{2}(z_{l-1}-z_l)^2\\
=& N_l - N_{l-1} - \tfrac{1}{2}\alpha^2(q^n+q^{-n})(q^l-q^{-l})^2\\
&+ \tfrac{1}{2}\alpha^2(q^l-q^{-l})^4\\
=& \frac{-\beta\gamma}{q(q-q^{-1})^2}(q^{2l}-q^{-2l})^2 - \alpha^2(q^l-q^{-l})^2.
\end{aligned}$$

Therefore the quadratic equation is

$$x^2 - \alpha(q^l-q^{-l})^2 x - \frac{q^{-1}\beta\gamma}{(q-q^{-1})^2}(q^{2l}-q^{-2l})^2 - \alpha^2(q^l-q^{-l})^2 = 0.$$

Note that for $n=1$ this equation equals $P_1(Y)=0$. It has the discriminant

$$\begin{aligned}
&\alpha^2(q^l-q^{-l})^4 + 4\left(\frac{q^{-1}\beta\gamma}{(q-q^{-1})^2}(q^{2l}-q^{-2l})^2 + \alpha^2(q^l-q^{-l})^2\right)\\
&= (q^{2l}-q^{-2l})^2\left(\alpha^2 + \frac{4q^{-1}\beta\gamma}{(q-q^{-1})^2}\right).
\end{aligned}$$

If R solves the equation

$$x^2 = \alpha^2 + \frac{4q^{-1}\beta\gamma}{(q-q^{-1})^2},$$

we get the zeros $\lambda_{n/2}$ and $\lambda_{-n/2}$, where

$$\lambda_r = \frac{\alpha}{2}(q^r - q^{-r})^2 + \frac{1}{2}(q^{2r} - q^{-2r})R \quad \text{for } 2r \in \mathbb{Z}.$$

For $n = 0$ this is the zero 0 of P_0. ∎

Remark 4.3: In [NM, Theorem 1] another method is sketched to get the zeros of $P_n(Y)$ if x acts diagonally. Starting from a matrix

$$g = \begin{pmatrix} \tilde{\alpha} & \tilde{\beta} \\ \tilde{\gamma} & \tilde{\delta} \end{pmatrix}$$

in $\mathrm{GL}(2;\mathbb{C})$ (but $\mathbb{C}$ can be replaced by k), the following element is considered:

$$D = -\tilde{\alpha}\tilde{\beta}E + (\tilde{\alpha}\tilde{\delta} + \tilde{\beta}\tilde{\gamma})\frac{K-1}{q-q^{-1}} + \tilde{\gamma}\tilde{\delta}q^{-1}FK$$

(the authors write q^H for K, $\sqrt{q}X_+q^{H/2}$ for E and $\sqrt{q}X_-q^{-H/2}$ for F). By construction of the eigenvectors of the left action of D on the quantum plane, it is shown that the distinct eigenvalues are

$$\lambda_m(g) = \frac{q^m - q^{-m}}{q - q^{-1}} \cdot (\tilde{\alpha}\tilde{\delta}q^m - \tilde{\beta}\tilde{\gamma}q^{-m})$$

where $2m \in \mathbb{Z}$, if $\tilde{\alpha}\tilde{\delta} - q^{2t}\tilde{\beta}\tilde{\gamma} \neq 0$ for all $t \in \mathbb{Z}$ (then D is called diagonalizable). This left action is represented by a matrix obtained from M_n by swapping the j-th column with the $(n+1-j)$-th column and the j-th row with the $(n+1-j)$-th row for $1 \leq j \leq (n+1)/2$ and therefore has the same characteristical polynomial as M_n, where

$$\alpha = \frac{\tilde{\alpha}\tilde{\delta} + \tilde{\beta}\tilde{\gamma}}{q - q^{-1}}, \quad \beta = q^{-1}\tilde{\gamma}\tilde{\delta}, \quad \gamma = -\tilde{\alpha}\tilde{\beta}.$$

Consequently, in the case $\tilde{\alpha}\tilde{\delta} - q^{2t}\tilde{\beta}\tilde{\gamma} \neq 0$ for all $t \in \mathbb{Z}$ there is the factorization

$$P_n(Y) = (\lambda_{n/2}(g) - Y)(\lambda_{n/2-1}(g) - Y)\cdots(\lambda_{-n/2}(g) - Y).$$

Up to the cases $D \in kE$ or $D \in kFK$, the products $\tilde{\alpha}\tilde{\delta}$ and $\tilde{\beta}\tilde{\gamma}$ do not vanish simultaneously, and if exactly one product vanishes, then D is always diagonalizable. Now assume that $\tilde{\alpha}\tilde{\beta}\tilde{-}\gamma\tilde{\delta} \neq 0$. In the factorization of $P_n(Y)$ both sides are polynomials in $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$, $\tilde{\delta}$. The equation is valid for an infinite number of values for $\tilde{\alpha}$ (when $\tilde{\beta}$, $\tilde{\gamma}$, $\tilde{\delta}$ are fixed), therefore it is an identity of polynomials in $\tilde{\alpha}$. A similar reasoning holds for $\tilde{\beta}$, $\tilde{\gamma}$, $\tilde{\delta}$. Thus $\lambda_{n/2}(g), \ldots, \lambda_{-n/2}(g)$ are the not necessarily distinct zeros of $P_n(Y)$ in any case, even if g is an arbitrary 2×2 matrix. Translating this into the notation of the Proposition, one gets the zeros λ_r considered in the Proposition. The condition $R \neq 0$ is equivalent to $\tilde{\alpha}\tilde{\delta} - \tilde{\beta}\tilde{\gamma} \neq 0$ (this is excluded by the choice of g in [NM], but also in this case D is skew primitive).

THEOREM 4.4: *Fix $R \in k$ such that $R^2 = \alpha^2 + 4q^{-1}\beta\gamma/(q - q^{-1})^2$. Then the following statements are equivalent:*

(1) *For all nonnegative integers n, the matrix M_n can be diagonalized.*
(2) *$R \neq 0$, and for all nonnegative integers m, the kernel of the action of x on $V_{1,2m}$ equals the generalized eigenspace for the eigenvalue 0 (i.e. if $v \cdot x^t = 0$ for some $t \geq 1$ and $v \in V_{1,2m}$, then $v \cdot x = 0$).*
(3) *There is no nonnegative integer n satisfying*

$$q^{-1}\beta\gamma\left(\frac{q^n + q^{-n}}{q - q^{-1}}\right)^2 = -\alpha^2.$$

Proof: The conclusion (1) $\Rightarrow$ (2) is trivial ($R = 0$ implies that $\lambda_r = \lambda_{-r}$ for all $r \in \frac{1}{2}\mathbb{Z} \setminus \{0\}$, and by the Lemma, the action of x on $V_{1,n}$ for $n > 1$ cannot be diagonalized).

In order to prove (2) $\Rightarrow$ (1), assume there is a polynomial P_n with a double zero. Then by 4.2 there are distinct half integers l and m such that $l + m$ is an integer (because $l, m \in I_n$) and $\lambda_l = \lambda_m$. Hence

$$\begin{aligned}
&\frac{\alpha}{2}(q^l - q^{-l})^2 + \frac{1}{2}(q^{2l} - q^{-2l})R = \\
= \quad &\frac{\alpha}{2}(q^m - q^{-m})^2 + \frac{1}{2}(q^{2m} - q^{-2m})R \\
\Rightarrow \quad &\frac{\alpha}{2}(q^{2l} + q^{-2l} - q^{2m} - q^{-2m}) + \frac{R}{2}(q^{2l} - q^{-2l} - q^{2m} + q^{-2m}) = 0 \\
\Rightarrow \quad &(q^{l-m} - q^{m-l})\left(\frac{\alpha}{2}(q^{l+m} - q^{-l-m}) + \frac{R}{2}(q^{l+m} + q^{-l-m})\right) = 0 \\
\Rightarrow \quad &(q^{l-m} - q^{m-l})\lambda_{l+m} = 0.
\end{aligned}$$

Since q is not a root of unity, this yields $\lambda_{l+m} = 0$. The assumption $R \neq 0$ implies $l + m \neq 0$, whence 0 is a double zero of $P_{2|l+m|}$, and by the Lemma, the generalized eigenspace of 0 does not equal the eigenspace in $V_{1,2|l+m|}$. Finally, condition (3) is equivalent to $R \neq 0$ and $\lambda_n\lambda_{-n} \neq 0$ for all positive integers n, i.e. to condition (2). ∎

There is a similar statement about $(1, K)$-primitive elements of the Hopf algebra $U_q(\mathrm{sl}(2))$.

Remark 4.5: 1. We consider $y := xK$. Then y is $(1, K)$-primitive and

$$\begin{aligned} M'_n &:= \rho_{e,n}(y) = \rho_{e,n}(K)M_n \\ &= \begin{pmatrix} (1-q^{-n})\alpha & q^{n-2}\gamma & 0 & \cdots & 0 \\ [n]\beta & (1-q^{2-n})\alpha & q^{n-4}[2]\gamma & \ddots & \vdots \\ 0 & [n-1]\beta & (1-q^{4-n})\alpha & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & q^{-n}[n]\gamma \\ 0 & \cdots & 0 & [1]\beta & (1-q^{n})\alpha \end{pmatrix}. \end{aligned}$$

Let M''_n be the matrix obtained from M'_n after replacing α by $-\alpha$ and interchanging β and γ. Comparison with M_n yields that for all $i, j \leq n+1$, the i, j-entry of M_n is the $(n+2-i, n+2-j)$-entry of M''_n, whence the characteristic polynomials of M_n and M''_n are identical. If $P_n(Y)$ is the characteristic polynomial of M''_n, then the characteristic polynomial of M'_n is $P_n(-Y)$ because all eigenvalues remain unchanged when β, γ are swapped, and replacing α by $-\alpha$ can be performed by multiplying the first, third, fifth, ... row and the second, fourth, sixth, ... column by -1. Thus 4.2(2) explicitly gives the zeros $-\lambda_r$ of the characteristic polynomial of the action of y on $V_{1,m}$ and two of them coincide if and only if this is true for x, too. An analog of 4.1 holds for y, too (the equation in part (2) becomes $(\xi z) \cdot y = \xi(z \cdot y)$), in particular all eigenspaces are one-dimensional. Therefore 4.4 can be proved analogously for y.

2. If there exists a nonnegative integer n such that

$$q^{-1}\beta\gamma\left(\frac{q^n + q^{-n}}{q - q^{-1}}\right)^2 = -\alpha^2,$$

then direct computation as in 4.4 gives $\lambda_r = \lambda_{n-r}$ for all r. If n is even then the eigenspace for $\lambda_{n/2}$ equals the generalized eigenspace and the eigenspaces for other eigenvalues are not the generalized eigenspaces. This explains why the case $R = 0$, where the eigenspace for the eigenvalue 0 is the generalized eigenspace but x is not diagonalizable, is not too special from this point of view. If n is odd then all eigenspaces are different from the generalized eigenspaces.

3. If and only if there is an element $n \in \frac{1}{2}\mathbb{N} \setminus \mathbb{N}$ satisfying

$$q^{-1}\beta\gamma\left(\frac{q^n + q^{-n}}{q - q^{-1}}\right)^2 = -\alpha^2,$$

then there is a non-zero element $\xi' \in k_q[a,b]_{2n}$ such that $\xi' \cdot x = 0$. This means that the algebras generated by 3 generators and 4 relations in analogy to Podleś' quantum spheres are proper right coideal subalgebras of the algebra $B = \{a \in A \mid x \cdot a = 0\}$ from the construction in 2.4, cf. [Mü]. These cases have been excluded in [KD] by the assumption that x should be $*$-invariant. The precise correspondence between the parameters α, β, γ used in this section and the parameter c (including the case $c = \infty$) in Podleś' original quantum spheres $S_{\mu c}$ (for $\mu = q$) is as follows:

$$c = \begin{cases} \frac{\beta\gamma q^{-1}}{\alpha^2(q-q^{-1})^2} & \text{if } \alpha \neq 0 \\ \infty & \text{if } \alpha = 0 \text{ (and } \beta\gamma \neq 0) \end{cases}$$

(in the $*$-invariant case we automatically have $\alpha \in \mathbb{R}$ and $q\bar{\gamma} = \beta$, whence

$$c = \left(\frac{|\gamma|}{\alpha(q - q^{-1})}\right)^2$$

is nonnegative). The special cases for negative c in [P] do not occur because then the finite dimensional quantum spheres cannot be canonically embedded into the function algebra as $*$-invariant subalgebras.

If x (or y respectively) is diagonalizable on all modules $V_{1,n}$, then the following general argument shows that it is diagonalizable on all modules $V_{e,n}$. Let kw be a one-dimensional $U_q(\mathrm{sl}(2))$ module with basis $\{w\}$, such that $E \cdot w = 0$, $F \cdot w = 0$, $K \cdot w = ew$. Then $V_{e,n} \cong kw \otimes V_{1,n} \cong V_{1,n} \otimes kw$, where $U_q(\mathrm{sl}(2))$ acts diagonally on the latter two modules and the second isomorphism is due to the Hopf algebra automorphism of $U_q(\mathrm{sl}(2))$, which maps E and F to $-E$ and $-F$ respectively and leaves K unchanged.

PROPOSITION 4.6: *Let U be a Hopf algebra, $g \in U$ a group-like element, $x \in U$ a $(g,1)$-primitive (respectively $(1,g)$-primitive) element, kw a one-dimensional left U-module with basis $\{w\}$ and V a finite dimensional left U-module. If the action of x on V is diagonalizable, then so is the diagonal action on $kw \otimes V$ (respectively $V \otimes kw$).*

Proof: Assume x is $(g,1)$-primitive. (The proof for $(1,g)$-primitive elements is similar.) Since kw is a one-dimensional U-module, there is a map $\chi\colon U \to k$ such that $u \cdot w = \chi(u)w$ for all $u \in U$. Let $v_1,\dots, v_n$ be a basis of V consisting of eigenvectors of the action of x, i.e. $x \cdot v_j = \lambda_j v_j$ for $j = 1,\dots, n$ and $\lambda_j \in k$. Then

$$\begin{aligned} x \cdot (w \otimes v_j) &= (g \cdot w) \otimes (x \cdot v_j) + (x \cdot w) \otimes v_j \\ &= \chi(g) w \otimes \lambda_j v_j + \chi(x) w \otimes v_j \\ &= (\chi(g)\lambda_j + \chi(x)) w \otimes v_j, \end{aligned}$$

whence the set $\{w\otimes v_1,\ldots, w\otimes v_n\}$ is a basis of $kw\otimes V$ consisting of eigenvectors of x. ∎

5. Application to $U_q(\mathfrak{g})$

As in the last section, we assume that k is an algebraically closed field of characteristic 0.

Let (a_{ij}) be an $n\times n$ matrix with integer coefficients such that $a_{ii}=2$ for all i, $a_{ij}\neq 0$ for all $i\neq j$, and there are relatively prime integers $d_1,\ldots,d_n\in\{1,2,3\}$ such that $(d_i a_{ij})$ is a symmetric positive definite matrix.

Thus (a_{ij}) is the Cartan matrix of a finite dimensional semisimple Lie algebra $\mathfrak{g}$.

Let $q\in k\setminus\{0\}$ be not a root of 1 and define $q_i := q^{d_i}$. The standard q-deformation $U_q(\mathfrak{g})$ is the algebra generated by E_i, F_i, K_i, K_i^{-1} subject to the following relations:

$$K_iK_i^{-1}=1=K_i^{-1}K_i,\quad K_iK_j=K_jK_i,$$
$$K_iE_jK_i^{-1}=q_i^{a_{ij}}E_j,\quad K_iF_jK_i^{-1}=q_i^{-a_{ij}}F_j,$$
$$E_iF_j-F_jE_i=\delta_{ij}\frac{K_i-K_i^{-1}}{q_i-q_i^{-1}}$$

for all i,j, and the q-deformed Serre relations (see [J, 5.1.1(vi)]) between the E_i's resp. the F_i's which we do not need explicitly. $U_q(\mathfrak{g})$ is a Hopf algebra where all the elements K_i are group-like and

$$\Delta(E_i)=1\otimes E_i+E_i\otimes K_i,\quad \Delta(F_i)=K_i^{-1}\otimes F_i+F_i\otimes 1$$

for all i. For $1\le i\le n$ let U_i be the subalgebra of $U_q(\mathfrak{g})$ generated by K_i, K_i^{-1}, E_i, F_i. Then $U_i\cong U_{q_i}(\mathrm{sl}(2))$ as Hopf algebras.

Let $\mathcal{C}$ be the semisimple tensor category of all finite dimensional left $U_q(\mathfrak{g})$-modules of type 1, that is all eigenvalues of the left multiplication with K_i for all i are of the form q^m, $m\in\mathbb{Z}$ (see for example [J, 4.3]). The dual Hopf algebra $U_q(\mathfrak{g})^0_{\mathcal{C}}$ is the q-deformed algebra of regular functions on the simply connected, connected semisimple algebraic group with Lie algebra $\mathfrak{g}$. Let $x\in U_q(\mathfrak{g})$ be a $(g,1)$-primitive element which is not a scalar multiple of $g-1$. Then there is some i such that $g=K_i^{-1}$ and $x\in U_i$ is a k-linear combination of $K_i^{-1}-1$, $E_iK_i^{-1}$ and F_i [CM, Theorem A]. We call x **semisimple** if multiplication with x is a diagonalizable operator on all finite dimensional left $U_q(\mathfrak{g})$-modules.

LEMMA 5.1: *For any $1\le i\le n$ let V be a finite dimensional left U_i-module of type 1 and $x\in U_i\subset U_q(\mathfrak{g})$ a $(K_i^{-1},1)$-primitive element.*

(1) *There exists a finite dimensional left* $U_q(\mathfrak{g})$*-module* W *of type 1 and a one-dimensional left* $U_q(\mathfrak{g})$*-module* ka *with* $E_j a = 0$, $F_j a = 0$ *and* $K_j a = \alpha_j a$ *where* $\alpha_j \in \{1, -1\}$ *for all* $1 \leq j \leq n$ *such that* V *is isomorphic to a* U_i*-submodule of* $ka \otimes W$ *and* $\alpha_i = 1$.

(2) x *is semisimple in* U_i *if and only if* x *is semisimple in* $U_q(\mathfrak{g})$.

Proof: (1) By [J, 10.1.14] or using the classification of highest weight modules of $U_q(\mathfrak{g})$, V is contained in a finite dimensional left $U_q(\mathfrak{g})$-module $\tilde{V}$. From the description of finite dimensional left $U_q(\mathfrak{g})$-modules it is known that $\tilde{V} \cong ka \otimes W$, where W is a left $U_q(\mathfrak{g})$-module of type 1 and $E_j a = 0$, $F_j a = 0$, $K_j a = \alpha_j a$, $\alpha_j \in \{1, -1\}$ for all j (see [J, 4.3]). It remains to show that $\alpha_i = 1$. Since V is of type 1, there is a non-zero $v \in V$ such that $K_i v = q^m v$ for some $m \in \mathbb{Z}$. Let $a \otimes w$ be the image of v in $ka \otimes W$. Then

$$q^m(a \otimes w) = K_i(a \otimes w) = K_i a \otimes K_i w = \alpha_i a \otimes K_i w,$$

hence $K_i w = \alpha_i^{-1} q^m w$. Since W is of type 1, we conclude $\alpha_i = 1$.

(2) If x is semisimple in U_i, then trivially x is semisimple as an element in $U_q(\mathfrak{g})$. Conversely, assume x is semisimple in $U_q(\mathfrak{g})$. Let V be any finite dimensional left U_i-module. By [J, 10.1.14] or the classification of highest weight modules of $U_q(\mathfrak{g})$, V is contained in a finite dimensional left $U_q(\mathfrak{g})$-module $\tilde{V}$. Hence multiplication with x is diagonalizable on $\tilde{V}$ and then on V, too. ∎

THEOREM 5.2: *Let* $1 \leq i \leq n$, $\alpha, \beta, \gamma \in k$, *and*

$$x = \alpha(K_i^{-1} - 1) + \beta K_i^{-1} E_i + \gamma F_i \in U_q(\mathfrak{g}) \setminus \{0\}$$

a $(K_i^{-1}, 1)$*-primitive element. Assume* $\alpha^2 + 4q^{-1}\beta\gamma/(q - q^{-1})^2 \neq 0$.

Let $\mathcal{C}$ *be the tensor category of finite dimensional left* $U_q(\mathfrak{g})$*-modules of type 1. Define* $A := U_q(\mathfrak{g})^0_{\mathcal{C}}$ *and* $B := \{a \in A \mid a \cdot x = 0\}$. *Then the following are equivalent:*

(1) x *is semisimple.*

(2) *There is no nonnegative integer* n *satisfying*

$$\alpha^2 + \beta\gamma q^{-1} \left(\frac{q^n + q^{-n}}{q - q^{-1}} \right)^2 = 0.$$

(3) A/AB^+ *is spanned by group-like elements.*

(3)′ A/B^+A *is spanned by group-like elements.*

(4) A *is faithfully flat as a left* B*-module.*

(4)′ *A is faithfully flat as a right B-module.*
(5) *B is a B-direct summand in A as a left B-module.*
(5)′ *B is a B-direct summand in A as a right B-module.*
(6) *B is simple in $\mathcal{M}_B^A$.*
(6)′ *B is simple in ${}_B\mathcal{M}^A$.*

If these conditions are satisfied, the set of characters $X(k[x],\mathcal{C})$ (cf. Corollary 2.4) is given by

$$X(k[x],\mathcal{C}) = \{\chi \in \mathrm{Alg}(k[x],k) \mid \chi(x) = \lambda_r, r \in \frac{1}{2}\mathbb{Z}\}$$

where the λ_r are the eigenvalues as computed in Theorem 4.2(2).

Proof: (1) $\iff$ (2): By 5.1(2) and section 3.

(1) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) follow from 2.5.

(4) $\Rightarrow$ (5) follows from 1.2.

(5) $\Rightarrow$ (6) is 1.3(2).

(6) $\Rightarrow$ (1): By 2.5(2), assumption (6) implies $B = \{a \in A \mid \exists n \geq 1\colon a \cdot x^n = 0\}$. Let W be any finite dimensional simple left $U_q(\mathfrak{g})$-module of type 1. By 2.1, part (2) and (3), $W \otimes W^* \cong C^W$ as right (and left) $U_q(\mathfrak{g})$-modules. Hence if $f \cdot x^n = 0$ for some $n \geq 1$ and $f \in W^*$, then $f \cdot x = 0$. Or equivalently, if $\phi\colon W^* \to W^*$ is right multiplication with x, then $\mathrm{Ker}(\phi) = \mathrm{Ker}(\phi^n)$ for all $n \geq 1$. If $\psi\colon W \to W$ is left multiplication with x, then $\psi^* = \phi$, and we get $\mathrm{im}(\psi) = \mathrm{im}(\psi^n)$, or equivalently, $\mathrm{Ker}(\psi) = \mathrm{Ker}(\psi^n)$ for all $n \geq 1$. Thus for all simple modules in $\mathcal{C}$, hence for all modules W in $\mathcal{C}$ (because $\mathcal{C}$ is semisimple), we have shown

$$\{w \in W \mid x \cdot w = 0\} = \{w \in W \mid \exists n \geq 1\colon x^n \cdot w = 0\}.$$

We want to show the same statement over U_i. Let V be a finite dimensional left U_i-module of type 1. By 5.1(1), V is isomorphic to a U_i-submodule of $ka \otimes W$, where W is a left $U_q(\mathfrak{g})$-module of type 1 and $E_i a = F_i a = 0$, $K_i a = a$. Hence

$$xa = \alpha(K_i^{-1} - 1)a + \beta K_i^{-1} E_i a + \gamma F_i a = 0,$$

and the action of x on any element $a \otimes w$, $w \in W$, is given by

$$\begin{aligned} x(a \otimes w) &= K_i^{-1} a \otimes xw + xa \otimes w, \quad \text{since } \Delta(x) = K_i^{-1} \otimes x + x \otimes 1 \\ &= a \otimes xw. \end{aligned}$$

In particular,

$$\{v \in V \mid x \cdot v = 0\} = \{v \in V \mid \exists n \geq 1\colon x^n \cdot v = 0\},$$

since this equality holds for W. Thus we see that condition (2) of Theorem 4.4 is satisfied (here we use the assumption $\alpha^2 + 4q^{-1}\beta\gamma/(q-q^{-1})^2 \neq 0$). Hence by 4.4(1), multiplication with x is diagonalizable on all $U_{q_i}(\mathrm{sl}(2))$-modules of type 1, and x is diagonalizable as an element in $U_{q_i}(\mathrm{sl}(2))$ by 4.6.

Define $y := xK_i$. Then y is $(1, K_i)$-primitive, and $B = \{a \mid a \cdot y = 0\}$. We now repeat the previous arguments with x replaced by y. Consider the statement

(1)′ y is semisimple .

We have shown in 4.5(1) and 4.6 that x is semisimple in U_i if and only if y is semisimple in U_i. Hence (1) $\iff$ (1)′ by 5.1(2).

(1)′ $\Rightarrow$ (3)′ and (3)′ $\Rightarrow$ (4)′ follow from 2.5.

(4)′ $\Rightarrow$ (5)′ follows from 1.2 (for A^{op}).

(5)′ $\Rightarrow$ (6)′ is 1.3(2) (for A^{op}).

(6)′ $\Rightarrow$ (2): By 2.5(2), $B = \{a \in A \mid \exists n \geq 1 \colon a \cdot y^n = 0\}$. Hence for all $W \in \mathcal{C}$,

$$\{w \in W \mid y \cdot w = 0\} = \{w \in W \mid \exists n \geq 1 \colon y^n \cdot w = 0\}.$$

Let V be a finite dimensional U_i-module of type 1. Then by 5.1(1), V is a U_i-submodule of $ka \otimes W$ for some left $U_q(\mathfrak{g})$-module W and $E_i \cdot a = 0$, $F_i \cdot a = 0$, $K_i \cdot a = a$. Since $ya = xa = 0$, $\Delta(y) = 1 \otimes y + y \otimes K_i$, and $\Delta(x) = K_i^{-1} \otimes x + x \otimes 1$, we have

$$y(a \otimes w) = a \otimes yw + ya \otimes K_i w = a \otimes yw.$$

Hence $\{v \in V \mid y \cdot v = 0\} = \{v \in V \mid \exists n \geq 1 \colon y^n \cdot v = 0\}$, since this holds for W. Therefore we get from 4.4 for y that (2) is satisfied.

The expression for $X(k[x], \mathcal{C})$ follows from the description of the eigenvalues of the action of x on simple $U_q(\mathrm{sl}(2))$ modules in 4.2. ∎

Remark 5.3:

1. Now assume $\alpha^2 + 4q^{-1}\beta\gamma/(q-q^{-1})^2 = 0$. This actually gives just one exception:
   - If $\alpha \neq 0$ consider $\tilde{x} = 2(K_i^{-1} - 1) + (q - q^{-1})K_i^{-1}E_i + (q^2 - 1)F_i$. Then

     $$B = \{a \in A \mid a \cdot x = 0\} = \{a \in A \mid a \cdot \tfrac{2}{\alpha} x = 0\}$$

     and the Hopf algebra automorphism of $U_q(\mathfrak{g})$

     $$(E_j, F_j, K_j) \mapsto \begin{cases} \left(\frac{2\beta}{(q-q^{-1})\alpha} E_i, \frac{2\gamma}{(q^2-1)\alpha} F_i, K_i\right) & \text{if } j = i \\ (E_j, F_j, K_j) & \text{if } j \neq i \end{cases}$$

     maps $\frac{2}{\alpha} x$ to $\tilde{x}$ and induces an isomorphism of B to the algebra $\{a \in A \mid a \cdot \tilde{x} = 0\}$ of infinitesimal invariants with respect to $\tilde{x}$.

- If $\alpha = \beta\gamma = 0$, then x is a scalar multiple of $K_i^{-1}E_i$ or F_i. Then x (and $K_i x$) acts nilpotently on all finite dimensional U_i-modules of type 1 and both x and $K_i x$ cannot be semisimple. Moreover, the numerical condition (2) does not hold. In this case 5.2 remains true.

(2) In the case $\mathfrak{g} = \mathrm{sl}_2$ the module structure is flat for any choice of α, β, γ. (It can be shown that A is the ascending union of free modules over B.)

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